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2000 J. Phys. A: Math. Gen. 33 6073

(http://iopscience.iop.org/0305-4470/33/35/301)

J. Phys. A: Math. Gen. **33** (2000) 6073–6080. Printed in the UK PII: S0305-4470(00)13717-5

Spreading of damage in a two-dimensional Ising model with dipolar interactions

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Received 3 May 2000

Abstract. In this work we use the spreading of damage technique to study the dynamical properties of a two-dimensional Ising model with competition between ferromagnetic exchange (J_0) and antiferromagnetic dipolar (J_d) interactions. In recent works it was shown that, depending on the ratio $\delta = J_0/J_d$, the ground state of the model corresponds to ferromagnetic stripes whose width increases with δ. In this striped phase the system displays a slow dynamics, characterized by the formation and growth of magnetic domains. We found that, within these phases, the spreading of damage technique allows one to identify two dynamical phases, with a behaviour similar to that observed in the Sherrington–Kirkpatrick model of spin glasses.

Magnetic models with competition between nearest-neighbour ferromagnetic and long-range antiferromagnetic dipolar interactions have been widely used during recent years in the modelling of many interesting magnetic phenomena. In particular, bi-dimensional systems in which the magnetic moments are aligned perpendicular to the plane have proven to be very useful in analysing ultrathin magnetic films, magnetic ordering of rare-earth subsystems of high-T_c superconductors and related layered compounds, avoided phase transitions in supercooled liquids and charge density waves in doped antiferromagnets, among others (see [1] and references therein). Perhaps the simplest way to take into account the interplay between ferromagnetic and antiferromagnetic ordering is to consider a bi-dimensional Ising model governed by the following Hamiltonian:

$$H = -J_0 \sum_{\langle i,j \rangle} \sigma_i \sigma_j + J_d \sum_{(i,j)} \frac{\sigma_i \sigma_j}{r_{ij}^3} \tag{1}$$

where the spin variable $\sigma_i = \pm 1$ is located at site i of a square lattice, the sum $\sum_{\langle i,j \rangle}$ runs only over all pairs of nearest-neighbour sites, while the sum $\sum_{(i,j)}$ runs over all distinct pairs of sites of the lattice and r_{ij} is the distance (in crystal units) between sites i and j. J_0 and $J_d > 0$ are the ferromagnetic exchange and antiferromagnetic dipolar coupling parameters respectively. For simplicity, we rewrite this Hamiltonian as follows:

$$H = -\delta \sum_{\langle i,j \rangle} \sigma_i \sigma_j + \sum_{(i,j)} \frac{\sigma_i \sigma_j}{r_{ij}^3} \tag{2}$$

with $\delta = J_0/J_d$.

In a recent work MacIsaac *et al* [1] studied the thermodynamics of the model and presented its finite-temperature phase diagram, obtained by means of a Monte Carlo simulation, which

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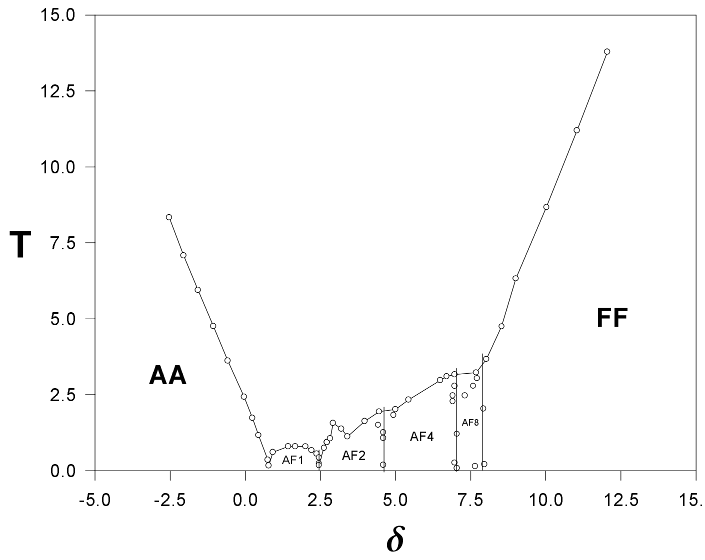

Figure 1. Phase diagram for a 16×16 system. The FF phase indicates that a single ferromagnetic stripe is observed due to the finite size of the simulated system.

we reproduce in figure 1. For $\delta < 0.85$, they found that the ground state of Hamiltonian (2) is the antiferromagnetic state. For $\delta > 0.85$ the antiferromagnetic state becomes unstable with respect to the formation of striped domain structures, that is, to state configurations with spins aligned along a particular axis, which form a ferromagnetic stripe of constant width h, so that spins in adjacent stripes are antialigned, resulting in a superlattice in the direction perpendicular to the stripes. The width h of the stripe increases with δ, and it is important to stress that the system never becomes ferromagnetic for finite values of δ [1] (an effect that cannot be observed in figure 1 due to the finite size of the samples).

Concerning its dynamics, the model is characterized by the formation and growth of magnetic domains, due to the competition between the exchange and dipolar interactions, which at low temperatures generate very large relaxation times. Sampaio *et al* [2] have shown the existence of two different regimes for the relaxation of the magnetization, depending on the value of δ. For $\delta > \delta_c \sim 2.7$ the magnetization relaxes exponentially, with a relaxation time that depends both on temperature and δ, while for $\delta < \delta_c$ the magnetization presents a power-law decay, with an exponent independent of δ. Toloza *et al* [3] have analysed the time evolution of the two-time auto-correlation function

$$C(t, t_w) = \frac{1}{N} \sum_i \langle \sigma_i(t + t_w)\sigma_i(t_w)\rangle \tag{3}$$

after the system has been quenched (from infinite temperature) into some non-equilibrium state, where $\langle \cdots \rangle$ means an average over different realizations of thermal noise and t_w is the waiting time, measured from the quenching time $t_0 = 0$. They found that the system presents ageing, i.e. a dependence on the previous history of the system: while for equilibrium states one expects that $C(t, t_w)$ depends on t and t_w only through the difference $t - t_w$, here it depends on both times, indicating that the system does not equilibrate in finite scales. Even more, for $0.85 < \delta < \delta_c \sim 2.7$ the function $C(t, t_w)$ obeys the dynamic scaling law

$$C(t, t_w) \sim c_\delta \frac{\ln(t)}{\ln[\tau(t_w)]} \tag{4}$$

as predicted by an activated scenario [4] proposed for spin glasses, while for $\delta > \delta_c$, where magnetization relaxes exponentially and the short-range ferromagnetic interactions dominate over the dipolar ones, $C(t, t_w)$ obeys the scaling law

$$C(t, t_w) \sim c_\delta \frac{t}{\tau(t_w)}. \tag{5}$$

Note that this behaviour is associated with an algebraic growth of the domain size $L(t) \propto t^\phi$ in systems with ferromagnetic ground state [5–7]. Through a study of the spin auto-correlation function and the conjugated response function to an external magnetic field, Stariolo and Cannas [8] have noted that this system violates the fluctuation–dissipation theorem.

In this paper we present a numerical study of the spreading of damage for the system described by Hamiltonian (2) (for a recent review of the technique see [9] and references therein). The main idea of the method consists in measuring the time dependence of the Hamming distance between two replicas with different initial conditions that evolve subjected to the same thermal noise (in numerical Monte Carlo simulations, both replicas are updated using the same random number sequence). Depending on whether the final Hamming distance between the two replicas is non-zero (the damage *spreads*) or vanishes (the damage *heals*), one can identify chaotic or non-chaotic behaviour, respectively.

Although it was shown that the spreading or healing of the Hamming distance depends not only on the intrinsic properties of the model (as one could have expected), but also on the specific algorithic implementation [10], the technique has proven to be very useful in determining critical exponents and critical temperatures. It has been observed that, by analysing the dependence of the long-time Hamming distance on the initial damage between replicas and on the temperature, one can identify dynamical critical temperatures separating different regimes. In particular, if a system is subjected to heat bath dynamics, one can associate a dynamical critical temperature with each static critical temperature, allowing a very accurate and efficient way to determine critical temperatures and critical exponents. (The opposite is not true, and generally one finds dynamical critical temperatures inside a static phase, i.e. not associated with a static transition [11, 12].)

We present the results of the long-time Hamming distance as a function of the temperature T for two values of the parameter δ: for $\delta = 2 < \delta_c$, where the magnetization decays as a power law and the ground state corresponds to stripes of width $h = 1$, and for $\delta = 4 > \delta_c$, where the magnetization decays exponentially and the ground state corresponds to stripes of width $h = 2$ (see figure 1).

We simulated systems of size $N = L \times L$, with L ranging from 16 to 40 (with particular emphasis on $L = 20$), with free boundary conditions. The system was subjected to a heat bath Monte Carlo dynamics, and time was measured in units of whole Monte Carlo sweeps over the lattice of N spins.

The procedure of the spreading of damage is as follows: for a given value of δ and T we let the system $\{\sigma_i^A\}$ evolve a transient time, and then we make a copy of the system, $\{\sigma_i^B\}$. We introduce *damage* in $\{\sigma_i^B\}$ by flipping a fraction of the spins. Given these two different initial conditions we let both systems evolve under the same thermal noise, i.e. by using the same random sequence to update both systems. We measure the time evolution of the Hamming distance between replicas, defined as

$$D(t) = \frac{1}{2N} \sum_i^N |\sigma_i^A(t) - \sigma_i^B(t)|. \tag{6}$$

In order to calculate a configurational average $\langle D(t) \rangle$ of the Hamming distance over initial states and random sequences we repeated the simulations for S different samples for each value of T and δ.

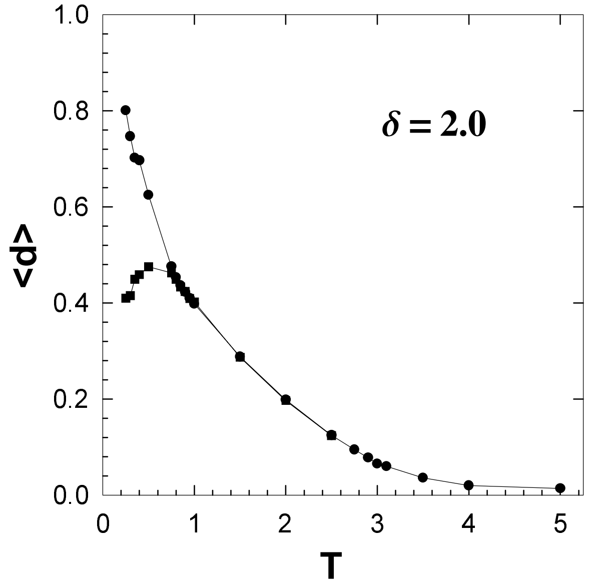

Figure 2. Averaged final Hamming distance $\langle d \rangle$ versus temperature, for $\delta = 2.0$, given two initial damages: $d(0) = 1.0$ (circles) and 0.35 (squares).

Note that, if at a given time both replicas become identical, they will remain identical for all subsequent times. This was taken into account by considering the fraction $P(t)$ of samples that did not become identical at time t, given by

$$P(t) = \lim_{S \to \infty} \frac{S_1(t)}{S} \tag{7}$$

where S_1 is the number of samples, from a total of S simulated samples, that are still different at time t. Using this *survival probability* we may write

$$\langle D(t) \rangle = \langle d(t) \rangle P(t) \tag{8}$$

where $\langle d(t) \rangle$ is the Hamming distance measured only over the $S_1(t)$ surviving samples at time t.

First we present the results for $\delta = 2.0$, where the ground state corresponds to stripes of width $h = 1$. In figure 2 we plot the averaged final Hamming distance $\langle d \rangle$ versus temperature T.

We can clearly identify two phases:

- in the low-temperature phase, for $T < T_d = 0.825 \pm 0.025$, the final value of $\langle d \rangle$ is non-zero, and it depends on the initial value of d;
- in the high-temperature phase, $T > T_d$, the final value of $\langle d \rangle$ is also always non-zero but its value is independent of the initial damage.

As already mentioned, the spreading of damage technique with heat bath dynamics has proven to be a very accurate method for determining thermodynamical critical temperatures [10]. This is due to the fact that damage transitions seem to agree with thermodynamical transitions. This becomes particularly relevant for those models whose thermostatics cannot be analytically solved and whose low-temperature phase has slow dynamics. In that case, it is very hard to determine transition temperatures by means of numerical simulations of equilibrium quantities.

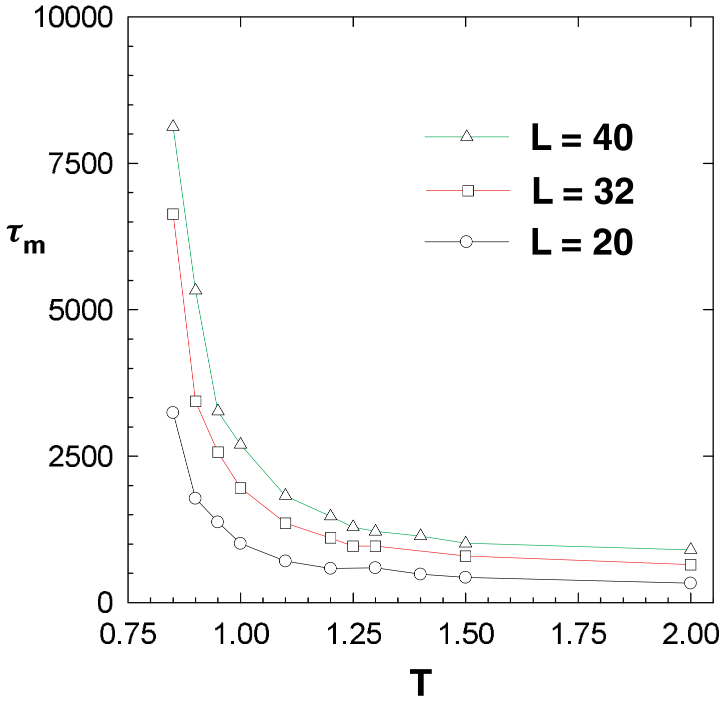

Figure 3. τ_m versus temperature, for $\delta = 2.0$, and for three different system sizes: $L = 20$, 32 and 40.

The model we are considering in this letter satisfies these two properties: very little is known about its thermodynamical phase diagram (as far as we known, the results of MacIsaac *et al* [1] reproduced in figure 1 are the most accurate calculations of the phase diagram) and its dynamics is a slow one, like those observed in glasses and coarsening (depending on δ). In order to determine with better accuracy the critical temperature of this model, we calculated the average time τ_m the system needs for the two replicas, which are initially in opposite configurations ($d(0) = 1$), to meet in phase space. One expects that this time increases with system size, diverging for $L \to \infty$, as one approaches the critical temperature T_d. In figure 3 we present the behaviour of $\tau_m(L)$ versus T (for $L = 20$, 32 and 40) in the high-temperature phase. We found that this quantity diverges at a critical temperature $T_d = 0.825 \pm 0.025$, which seems to agree with that observed in [1].

Observing the temporal behaviour of the final Hamming distance $\langle D(t) \rangle$ in the high-temperature phase, one can identify two different regimes, as can be seen in figure 4, where we plot $\langle D(t) \rangle$ versus t for two different temperatures, namely, 1.0 and 3.5 (both above T_d) and two different initial damages, $D(0) = 1.0$ and 0.35, for a system of linear size $L = 20$.

For $T = 3.5$ the behaviour could be associated with a dynamical phase with null value of the final Hamming distance, as usually occurs in paramagnetic phases at high enough temperatures. Nevertheless, a very careful finite-size study reveals a very different behaviour. In figure 5 we present the fraction of surviving replicas $P(t)$, the Hamming distance $\langle D(t) \rangle$ and also $\langle d(t) \rangle$ versus t, for $T = 3.5$ when the initial damage is $D(0) = 1.0$ and $N = 20 \times 20$. The results presented correspond to $S = 500$ samples.

For short times $P(t)$ remains always equal to one while $\langle D(t) \rangle$ decays. This behaviour can be associated with the formation of basins in phase space separated by energy barriers that

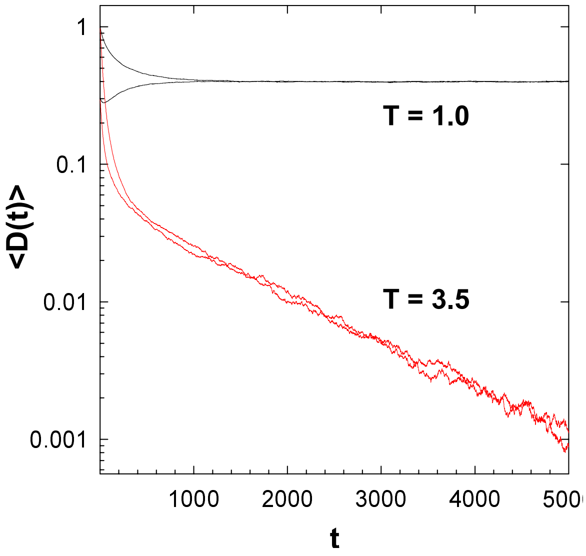

Figure 4. Average Hamming distance $\langle D(t)\rangle$ when $\delta = 2.0$, for temperatures $T = 1.0$ and 3.5 with two different initial damages: $D(0) = 1.0$ and 0.35, for a system with linear size $L = 20$.

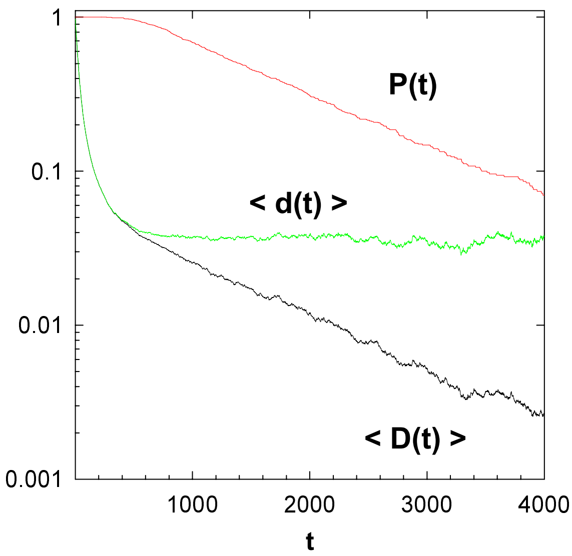

Figure 5. Fraction of surviving replicas $P(t)$, Hamming distances $\langle D(t)\rangle$ and $\langle d(t)\rangle$ for $T = 3.5$, when $D(0) = 1.0$, for a system with linear size $L = 20$.

avoid the two replicas meeting. After this transient (that depends on the system size N), the system enters a second regime in which $P(t)$ and $\langle D\rangle$ decay exponentially in such a way that $\langle d\rangle$ *remains constant*, indicating that the surviving replicas are always separated by the same distance in phase space.

In this regime the system is very sensitive to finite-size effects, as can be seen in figure 6, where we present the variation of $P(t)$ at $T = 3.5$ as system size increases ($L = 16, 20$ and 32).

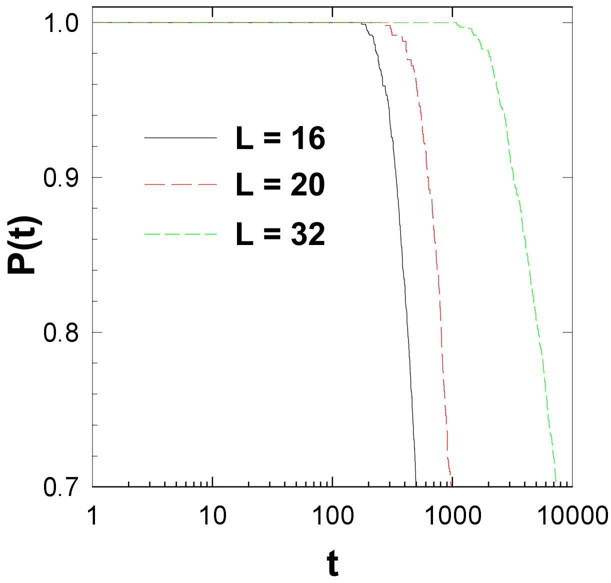

Figure 6. Fraction of surviving replicas $P(t)$ versus time for three different system sizes: $L = 16$, 20 and 32.

We can conclude that, in the thermodynamical limit ($L \to \infty$), the survival probability will remain always equal to one, indicating that the adequate quantity to characterize the damage is $\langle d \rangle$, which does not depend on N.

It is important to stress that, as far as we know, the unique magnetic model that displays a similar behaviour is the Sherrington–Kirkpatrick model, for which it has been found that $d(T)$ decreases as T^{-2}, keeping a non-zero value for all finite temperatures.

A similar qualitative behaviour of $D(t)$, $P(t)$ and $d(t)$ was observed for $\delta = 4.0$. In figure 7 we present the behaviour of $\langle d(t) \rangle$ as a function of temperature, and again we observe two different regimes, now with critical temperature $T_c = 2.225 \pm 0.025$. Note that, for this value of δ, the dynamics corresponds to that of a coarsening process, while for the former case $\delta = 2$ the dynamics is similar to that of a short-range spin glass. Nevertheless, as occurred when the violation of the fluctuation–dissipation theorem was studied [8], the spreading of damage technique does not reveal any sensitivity to this difference.

We have studied the behaviour of the final Hamming distance in two different dynamical phases: $\delta = 2.0$ and 4.0. When $\delta = 2.0$ in the low-temperature phase, the dynamics is characterized by a power law decay of the magnetization and an aging behaviour characteristic of spin glasses, while for $\delta = 4.0$ also in the low-temperature phase, the magnetization decays exponentially and the dynamics is governed by a coarsening process. We found that the spreading of damage technique, applied to system (2) subjected to a heat bath dynamics (remember that this technique is very sensitive to the stochastic dynamics implemented), reveals a dynamical phase transition that seems to agree with that separating the high-temperature paramagnetic phase from the low-temperature striped phase. Concerning the paramagnetic phase, we observed that its dynamical behaviour is similar to that observed in the SK model, where it always has a non-zero value that is independent of the initial Hamming distance. This result is very unusual in magnetic systems, and we believe it could be associated with the long-range nature of the interactions. In the low-temperature phase (striped phase) the damage

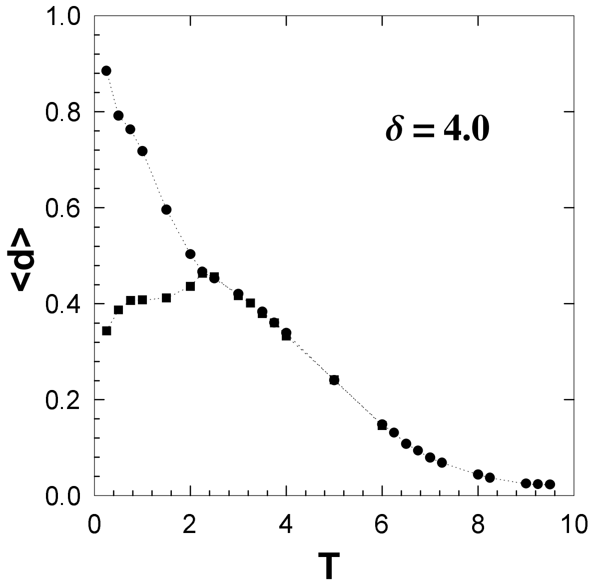

Figure 7. Averaged final Hamming distance $\langle d \rangle$ versus temperature, for $\delta = 4.0$, given two initial damages: $d(0) = 1.0$ (circles) and 0.35 (squares).

displays a strong dependence on the initial damage, but we could not identify any sensitivity of the technique to distinguish between glassy and coarsening dynamics.

Acknowledgments

The authors would like to thank Julio H Toloza for helpful discussions. This work was partially supported by grants from Consejo Nacional de Investigaciones Científicas y Técnicas Conicet (Argentina), Consejo Provincial de Investigaciones Científicas y Tecnológicas Conicor (Córdoba, Argentina) and Secretaría de Ciencia y Tecnología de la Universidad Nacional de Córdoba Secyt (Córdoba, Argentina).

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